

- (b) Assuming at $\frac{y}{h} \ll 1$, there is a region where the viscous forces are negligible, then from (6.69), we have (for $R_\tau \rightarrow \infty$)

$$-\overline{u'v'} = u_\tau^2 \quad (6.74)$$

Consider Prandtl's mixing length hypothesis

$$-\overline{u'v'} = l_m^2 \left(\frac{d\bar{u}}{dy} \right)^2 \quad (6.75)$$

where the mixing length is

$$l_m = \chi y \quad (6.76)$$

The parameter χ is called von-Karman constant.

Here, Karman assumed that l_m should be a function of the distance from the wall in a wall bounded turbulent flow, rather than a constant as taken by Prandtl in the case of free turbulent flows. Substituting Eqs. (6.75) and (6.76) into Eq. (6.74), we obtain

$$\chi^2 y^2 \left(\frac{d\bar{u}}{dy} \right)^2 = u_\tau^2 \quad (6.77)$$

$$\chi y \frac{d\bar{u}}{dy} = u_\tau \quad (6.78)$$

$$\chi \frac{du^+}{dy^+} = \frac{1}{y^+} \quad (6.79)$$

Integrating Eq. (6.79) between 0 and y^+ , we get

$$u^+ = \frac{1}{\chi} \ln y^+ + C \quad (6.80)$$

where, $\chi \approx 0.4$ and $C \approx 5.0$.

Equation (6.80) is called the Logarithmic Law of the Wall, which is valid for $30 \leq y^+ \leq 140$, the inertial sublayer.

6.7 TURBULENT HEAT TRANSFER IN PIPE (SIMPLIFIED ANALYSIS)

Heat flux due to conductivity of a fluid (Fig. 6.9) is:

$$q_l'' = -k \frac{\partial \bar{T}}{\partial y} \quad (6.81)$$

The mean rate of turbulent heat transfer is given by

$$q_t'' = \rho c_p \overline{v'T'} \quad \left[c_p \overline{u'T'} < c_p \overline{v'T'} \right] \quad (6.82)$$

The total heat flux is given by

$$q'' = q_l'' + q_t'' = -k \frac{\partial \bar{T}}{\partial y} + \rho c_p \overline{v'T'} \quad (6.83)$$

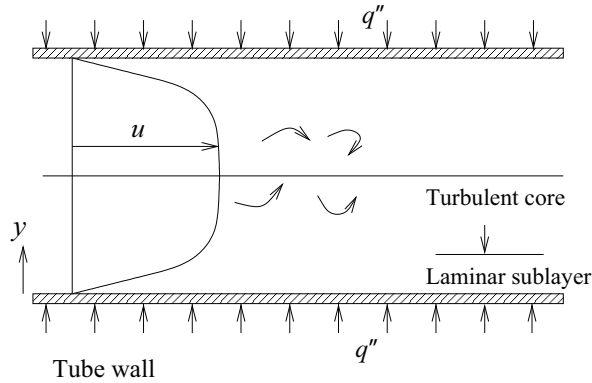


Figure 6.9 Viscous sublayer and turbulent core in a pipe flow

or

$$q'' = -\rho c_p \left[\alpha \frac{\partial \bar{T}}{\partial y} - \overline{v'T'} \right] \quad (6.84)$$

Turbulent heat exchange is also considered equivalent to an increase in thermal diffusivity of the fluid similar to turbulent momentum exchange

$$\overline{v'T'} = -\alpha_t \frac{\partial \bar{T}}{\partial y} \quad (6.85)$$

$$q'' = -\rho c_p [\alpha + \alpha_t] \frac{\partial \bar{T}}{\partial y} \quad (6.86)$$

where α_t is eddy diffusivity.

If q is the total heat transfer, then q may be written as

$$\frac{q}{\rho c_p A} = -(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial y} \quad (6.87)$$

In a similar way, shear stress in a turbulent flow could be written as

$$\frac{\tau}{\rho} = (\nu + \nu_t) \frac{\partial \bar{u}}{\partial y} \quad (6.88)$$

We rely on the Chilton-Colburn analogy at this stage. The analogy says that the heat and momentum are transported at the same rate; that is, $\nu = \alpha$ (Prandtl number, $\frac{\nu}{\alpha} = 1$) and $\nu_t = \alpha_t$ (turbulent Prandtl number, $\sigma_t = 1$). Dividing equation Eq. (6.87) by Eq. (6.88) we get

$$\frac{q}{c_p A \tau} d\bar{u} = -d\bar{T} \quad (6.89)$$

or

$$(q/c_p A \tau) d\bar{u} = -d\bar{T}$$

An additional assumption is invoked here; ratio of heat transfer per unit area to shear stress is constant across the flow field. Thus,

$$\frac{q}{A\tau} = \text{constant} = \frac{q_w}{A_w\tau_w} \quad (6.90)$$

Now Eq. (6.89) can be integrated between $y = y_1$ and $y = y_2$

$$\frac{q_w}{c_p A_w \tau_w} \int_{y=y_1}^{y=y_2} d\bar{u} = \int_{y=y_1}^{y=y_2} -d\bar{T}$$

One of the limits of integration may be chosen as $y_1 =$ conditions prevalent at the wall. The other limit $y = y_2$ is chosen in such a way that $\bar{T}(y_2)$ is equal to the bulk mean temperature T_m and $\bar{u}(y_2)$ is approximately the same as the mean velocity u_m

$$\frac{q_w}{c_p A_w \tau_w} \int_{\bar{u}=0}^{\bar{u}=u_m} d\bar{u} = \int_{\bar{T}=T_w}^{\bar{T}=T_m} -d\bar{T} \quad (6.91)$$

or

$$\frac{q_w u_m}{A_w \tau_w c_p} = T_w - T_m \quad (6.92)$$

We know

$$q_w = h A_w (T_w - T_m) \quad (6.93)$$

From Fig. 6.10, one can obtain

$$\tau_w = \frac{\Delta p (\pi D^2)}{4\pi D L} = \frac{\Delta p}{4} \frac{D}{L} \quad (6.94)$$

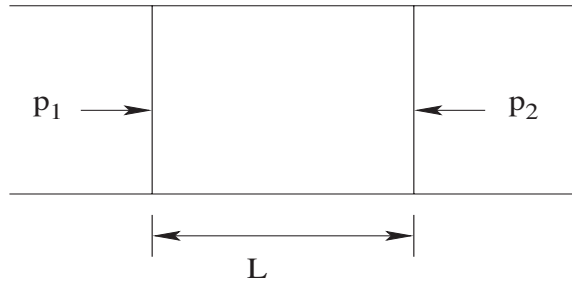


Figure 6.10 Pressure differential in a channel

The pressure drop in a pipe flow can be written as

$$\frac{\Delta p}{\rho} = \frac{f L u_m^2}{2D}$$

Invoking the pressure drop expression in Eq. (6.94) we get

$$\tau_w = \frac{\rho f L u_m^2}{4(2D)} \frac{D}{L} = \frac{f}{8} \rho u_m^2 \quad (6.95)$$

Invoking Eqs. (6.93) and (6.95) in Eq. (6.92) we get

$$\frac{8 h A_w (T_w - T_m) u_m}{A_w c_p \rho u_m^2 f} = (T_w - T_m) \quad (6.96)$$

or

$$\frac{h}{\rho c_p u_m} = \frac{f}{8} \quad (6.97)$$

or

$$St = \frac{Nu_D}{Re_D Pr} = \frac{f}{8} \quad (6.98)$$

Turbulent friction factor may be assumed as $f = 0.184 Re_D^{-1/5}$ for $Re \geq 2 \times 10^4$

$$\frac{Nu_D}{Re_D Pr} = \frac{0.184 Re_D^{-1/5}}{8}$$

or

$$Nu_D = 0.023 Re^{4/5} \quad (6.99)$$

This relation of turbulent heat transfer is highly restrictive because of the $Pr = 1.0$ assumption. Let us return to the Chilton-Colburn analogy for energy and momentum transport. The analogy holds good for $Pr \geq 0.5$ and we can thus write

$$St Pr^{2/3} = \frac{C_f}{2} \quad (6.100)$$

Again we know that $C_f = f/4$

$$St Pr^{2/3} = \frac{f}{8} \quad (6.101)$$

Now this dependence on Prandtl number works well for turbulent pipe flow. Accordingly Eq. (6.98) can be modified as

$$St Pr^{2/3} = \frac{f}{8} \quad \text{with friction factor } f = 0.184 Re_D^{-1/5}$$

or

$$\frac{Nu_D}{Re_D Pr} Pr^{2/3} = \frac{0.184 Re_D^{-1/5}}{8}$$

or

$$Nu_D = 0.023 [Re_D]^{4/5} [Pr]^{1/3} \quad (6.102)$$

Dittus and Boelter [2] established experimental correlation which is quite close to the above equation. The correlation is:

$$Nu_D = 0.023 (Re_D)^{4/5} (Pr)^n \quad (6.103)$$

$n = 0.4$ for heating, i.e., $T_w > T_m$ and $n = 0.3$ for cooling, i.e., $T_w < T_m$. The result has been confirmed experimentally for the range of conditions: $0.7 \leq Pr \leq$

160, $Re_D \geq 10000$ and $L/D \geq 10$. All fluid properties are evaluated at arithmetic mean of bulk temperature, i.e., $\bar{T}_m = (T_{m,o} + T_{m,i})/2$. For flows with large property variations of the fluids, the correlation due to Sieder and Tate [3] is recommended.

$$Nu_D = 0.027 Re_D^{4/5} Pr^{1/3} \left[\frac{\mu}{\mu_w} \right]^{0.14} \quad (6.104)$$

The range of applicability is: $0.7 \leq Pr \leq 16700$, $Re_D \geq 10000$, and $L/D \geq 10$ where all the properties except μ_w are evaluated at $\bar{T}_m$. Note that, to a good degree of approximation, the correlations may be applied for both the uniform wall temperature and uniform wall heat flux conditions. The μ_w is the property of the fluid, to be evaluated at the wall temperature. The foregoing correlations are restricted to smooth surface conditions. Heat transfer will be enhanced by surface roughness. In fact, heat transfer enhancement is often promoted by using internal ribs, twisted tapes, fins or corrugations.

The above mentioned correlations do not apply to heat transfer in liquid metals ($3 \times 10^{-3} \leq Pr \leq 5 \times 10^{-2}$). For fully developed turbulent flow ($L/D \geq 10$), in smooth circular tubes with uniform wall heat flux, the recommended correlation for liquid metals (due to Skupinski, Tortel and Vautrey [4]) is

$$Nu_D = 4.82 + 0.0185 (Pe_D)^{0.827} (q_w'' = \text{uniform})$$

Similarly for uniform wall temperature, the correlation (due to Seban and Shimazaki [5]) is

$$Nu_D = 5.0 + 0.025 (Pe_D)^{0.8} (T_w = \text{uniform})$$

6.8 COMPUTATIONAL APPROACH AND k - ϵ MODEL OF TURBULENCE

The time averaged governing equations for mass, momentum and energy may be written as

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (6.105)$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial \bar{u}_i}{\partial x_j} - \overline{u'_i u'_j} \right) \quad (6.106)$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_i \frac{\partial \bar{T}}{\partial x_i} = \frac{\partial}{\partial x_i} \left[\alpha \frac{\partial \bar{T}}{\partial x_i} - \overline{u'_i T'} \right] \quad (6.107)$$

with

$$-\overline{u'_i u'_j} = \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \quad (6.108)$$

$$-\overline{u'_i T'} = \alpha_t \frac{\partial \bar{T}}{\partial x_i} \quad (6.109)$$

and

$$\alpha_t = \nu_t / \sigma_t \quad (6.110)$$

where σ_t is turbulent Prandtl number. The entire system of equations can be closed if σ_t is known and ν_t is determined correctly.

We can construct an expression for turbulent viscosity or eddy viscosity in terms of two attributes of an eddy. The eddy viscosity can be viewed as a product of velocity scale and a length scale. The velocity scale is

$$v = k^{1/2}$$

and the length scale can be shown to be

$$l = k^{3/2}/\epsilon$$

Thus

$$\nu_t = C_\mu \frac{k^2}{\epsilon} \quad (6.111)$$

The quantity C_μ is a model constant to be determined accurately for a specific flow. The kinetic energy k and its dissipation rate ϵ are evaluated at each point in the domain from their governing differential equations. The equation for k can be derived from the Navier-Stokes equations by subtracting the mean equations from the unaveraged equations to obtain an equation for the fluctuating velocity (say u'_i). Taking the scalar product of this equation with the fluctuating velocity (u'_j) and averaging yields the equation for the Reynolds stress transport. The transport equation for kinetic energy can be obtained by substituting $j = i$ in the Reynolds stress transport equation. An equation for the dissipation can be derived from the Navier-Stokes equations by an extension of the method deployed to derive the kinetic energy equation.

The governing equation for k is given by

$$\begin{aligned} & \underbrace{\frac{\partial k}{\partial t}}_{(1)} + \underbrace{\overline{u_j} \frac{\partial k}{\partial x_j}}_{(2)} = - \underbrace{\tau_{ij} \frac{\partial \overline{u_i}}{\partial x_j}}_{(3)} - \underbrace{\epsilon}_{(4)} \\ & - \underbrace{\frac{\partial}{\partial x_j} \left(\frac{1}{2} \overline{u'_i u'_i u'_j} + \frac{1}{\rho} \overline{p' u'_j} \right)}_{(5)} + \underbrace{\nu \nabla^2 k}_{(6)} \end{aligned} \quad (6.112)$$

In Eq. (6.112), term-1 is the rate of change of turbulent kinetic energy k , term-2 is the convective transport of k , term-3 is the production by shear, term-4 is the viscous dissipation and term-5 is the third-order turbulent diffusive transport of k . The term-6 is the viscous diffusion of k . The production term represents the transfer of kinetic energy from the mean flow to the turbulent motion. τ_{ij} is the turbulent shear stress.

A scalar dissipation rate ϵ is defined as

$$\epsilon = \frac{1}{2} \epsilon_{ii} = \frac{1}{2} 2\nu \overline{\left(\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} \right)} \quad (6.113)$$

Instead of solving Eq.(6.112), the turbulent kinetic energy, k is obtained by solving the modeled transport equations.

The modeled transport equation for k is

$$\begin{aligned} \frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} &= \nu_t \left[\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right] \frac{\partial \bar{u}_i}{\partial x_j} \\ &+ \frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_j} \right) - \epsilon + \nu \nabla^2 k \end{aligned} \quad (6.114)$$

The first term on the right hand side corresponds to production of turbulent kinetic energy. The second term on the right hand side signifies turbulent diffusion of turbulent kinetic energy.

The transport equation for ϵ is also a modeled version of the original transport equation for ϵ . Original transport equation is not being mentioned herein.

The modeled equation for ϵ is

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_i \frac{\partial \epsilon}{\partial x_i} = \nu \nabla^2 \epsilon + P_\epsilon + D_\epsilon - \phi_\epsilon \quad (6.115)$$

As to symbols, P_ϵ , D_ϵ and ϕ_ϵ represent the production of dissipation, the turbulent diffusion of dissipation and the turbulent destruction of dissipation respectively. The diffusion term, D_ϵ can be modeled using the gradient transport (similar to the modeled k -equation) hypothesis as

$$D_\epsilon = \frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right) \quad (6.116)$$

where σ_ϵ is an empirical constant.

The destruction of dissipation is assumed to depend only on the length and the time scales. This can be written as

$$\phi_\epsilon = C_{\epsilon 2} \frac{\epsilon^2}{k} \quad (6.117)$$

where $C_{\epsilon 2}$ is an empirical constant.

The production of dissipation is assumed to be proportional to the production of the turbulent kinetic energy, the length scale and the time scale of turbulence. A simple dimensional analysis for the production term along with the above assumption leads to

$$\begin{aligned} \frac{\partial \epsilon}{\partial t} + \bar{u}_j \frac{\partial \epsilon}{\partial x_j} &= \nu \nabla^2 \epsilon + \frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right) \\ &+ C_{\epsilon 1} \frac{\epsilon}{k} \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j} - C_{\epsilon 2} \frac{\epsilon^2}{k} \end{aligned} \quad (6.118)$$

Equation (6.118) is the final expression for the modeled version of the transport equation for the turbulent dissipation rate.

The standard k - ϵ model has five empirical constants C_μ , σ_k , σ_ϵ , $C_{\epsilon 1}$ and $C_{\epsilon 2}$ in its formulation. $C_{\epsilon 2}$ is determined using the experiments on the decay of k behind a grid and a value of $C_{\epsilon 2} = 1.92$ is widely accepted. The experiments on the local-equilibrium-shear-layers suggested that $C_\mu \approx 0.09$. The constants σ_k , σ_ϵ and $C_{\epsilon 1}$ were determined and their suggested values are 1.0, 1.3 and 1.44 respectively. It